Working With Rotations

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1 Introduction

Rotations in 3D space are useful for many applications — physics engines, robotics, perception — as they encode the orientation of a rigid body. However, they are tricky: they belong to a group[1] but not a vector space, i.e. the linear combination of rotations is not a rotation. As a consequence, the usual algorithms for interpolation, integration or optimization won't work out of the box with rotations, or will work imperfectly.

Various sets of parameters — or *parameterizations* — can be used to describe rotation objects. Geometrically, a rotation is characterized by its axis and a scalar angle. Hence, intuitively, a rotation has three degrees of freedom: two for the direction of its axis, and one for its angle. However, since rotations do not form a vector space, there exists no ideal parameterization, i.e. a differentiable bijection between rotations and a set of parameters belonging to $\mathbb{R}^3$ (this would be ideal because usual algorithms on vector spaces could be applied to rotations without modification). There is therefore no single best choice of parameters for working with rotations. It has to be adapted to the application and the context.

The choice of a parameterization for rotations is a trade-off between *redundancy* and *singularity*. As we will see, all common parameterizations lie in a vector space of dimension three or greater. Minimal parameterizations have exactly three independent vector space parameters, but have singularities, intuitively defined as a lack of injectivity: there is a non-trivial subspace of the parameter space that yields the same rotation. Redundant parameterizations have strictly more than three independent vector space parameters, which allow them to be singularity-free. The price to pay for this redundancy is that these parameters must meet certain constraints. On the one hand, at a singularity point, algorithms will usually break in various ways. On the other hand, it is not obvious to meet parameter constraints while applying a given algorithm: often, it changes its complexity[2]. Hence, for a given application, the choice of a parameter set for rotations can usually be guided by answering the following questions:

- Can singularities be avoided? It is usually possible if the rotations involved in the problem at hand only span a subset of all rotations, or if one can re-parameterize one's way out of singularities.
- Can parameter constraints be applied together with the target algorithm? Constraints may be applied exactly or loosely[3]
- Is the parameter choice efficient for the rotation computations which need to be performed by the algorithm? Different parameter sets have different performance trade-offs with respect to elementary computations such as composition, vector rotation, integration or Jacobian computations for optimization.

[1] They are the simplest real-life example of a *Lie group*

[2] For instance, constrained optimization is more complex than unconstrained optimization

[3] The simplest solution is to ignore constraints and apply the algorithm *as if the parameters were in a vector space*, followed by a projection step on the constraint set. One of the simplest example of this strategy is normalized quaternion interpolation LERP, also known as NLERP.

The most widely used parameterizations are 3×3 matrices with orthonormality constraints, Euler angles, and unitary quaternions. However, many other possible parameterizations exist. While some are merely mathematical curiosities, others can be very useful, depending on the application. Amongst these are exponential coordinates and MRP — Modified Rodrigues Parameters —, a lesser-known but very interesting parameter set. The main purpose of this paper is to present, in a consistent style, a self-contained calculation of the Lie differentials of both the exponential coordinates and the MRP parameterization, and their respective inverses. For this reason, we chose to carry out all computations using only elementary calculus, and not Lie group theory.

First, in section 2, we recall basic useful 3D rotation formulas and their connections with unitary quaternions. Since this material is widely available elsewhere, we don't go into much detail. Section 3 defines left and right Lie differentials — a.k.a. Jacobians — for smooth functions taking their values in the space of unitary quaternions Q_u, which is a Lie group. We then connect these seemingly artificial definitions with the time derivative — i.e. angular velocity — of a parameterized rotation curve. In contrast with most references on the subject ([1], [2]), we don't work directly with parameterizations taking their value in the group SO_3 of 3D rotations, but instead use unitary quaternions as the arrival space. This means there is a small discrepancy in the Jacobian formulas — specifically, by a factor of 2 — with these references. This is not a problem however, as this discrepancy is resolved when using the Lie differentials to compute actual angular velocities. Section 4 then discusses and derives the Lie differentials of the exponential parameterization and its inverse (the logarithm map), while section 5 discusses and derives the Lie differentials of the MRP parameterization and its inverse.

2 3D Rotation Matrices and Unitary Quaternions

This section recalls well-known formulas concerning the group of 3D rotation matrices SO_3, and its connection with the group of unitary quaternions Q_u. Since this material is widely available elsewhere, we will skip derivation details.

2.1 Matrices

2.1.1 Definition

Rotations in 3D space can be represented by 3×3 matrices R satisfying the following property:

$$R^T R = R R^T = \mathrm{Id}, \quad \det(R) = 1. \tag{1}$$

2.2 Quaternions

2.2.1 Definition

We assume some familiarity with the representation of 3D rotations as unitary quaternions, and will just recall some formulas we will need later on. Given a unitary quaternion

q, we denote by q_s and q_v its scalar and vector part:

$$q = (q_s, \mathbf{q_v}), \quad q_s \in \mathbb{R}, \quad \mathbf{q_v} \in \mathbb{R}^3. \tag{2}$$

The quaternion q being normalized, we have

$$q_s^2 + \|q_v\|^2 = 1. \tag{3}$$

the *quaternion product* $*$ is defined by the following formula:

$$q_1 * q_2 = (q_{s_1} q_{s_2} - \mathbf{q_{v_1}} \cdot \mathbf{q_{v_2}}, \, q_{s_1} \mathbf{q_{v_2}} + q_{s_2} \mathbf{q_{v_1}} + \mathbf{q_{v_1}} \times \mathbf{q_{v_2}}) \, . \tag{4}$$

A rotation of — normalized — axis $\mathbf{n}$ and angle θ can be represented by the following — normalized — quaternion:

$$q = \left(\cos\left[\tfrac{\theta}{2}\right], \, \sin\left[\tfrac{\theta}{2}\right] \mathbf{n}\right). \tag{5}$$

All in all, quaternions are an interesting balanced parameterization of rotations: they have no singularities, yet have few parameters (only one parameter more than a minimal parameterization). Moreover, the constraint which needs to be satisfied by a quaternion parameterization — the unitary constraint — is rather simple. This makes quaternions a good and safe default choice. Nevertheless, if one is able to avoid singularities, using minimal parameterizations can be worth the extra effort (see section 1).

2.3 Euler Angles

The best-known family of minimal rotation parameterizations is the so-called Euler angles. We will skip results for Euler angles as relevant material is widely available elsewhere. The main drawback of using Euler angles is that singularities — called *gimbal locks* in this context — can occur relatively easily. For this reason, we will focus on two alternative minimal parameterizations: exponential coordinates and Modified Rodrigues Parameters — MRPs. But first, in the next section, we will define the basic tool we need to carry out computations: Lie differentials.

3 Left and Right Lie Differentials for Quaternion-Valued Functions

3.1 Definition

We introduce Lie differentials somewhat out of the blue in the following definition. We then explain why we care about these objects in section 3.2.

Definition 1 *Let $\varphi : \mathbb{R}^n \to Q_u$ be a function whose values are unitary quaternions. Let us define $\tilde{\varphi}$ as the same function where the target space is considered to be $\mathbb{R}^4$, and suppose that $\tilde{\varphi}$ is differentiable. For a given point $x \in \mathbb{R}^n$, we define the left — resp.,* *right — Lie differential of* φ *at point* x $\mathrm{D}^l_x\varphi : \mathbb{R}^n \to \mathbb{R}^3$ *— resp.,* $\mathrm{D}^r_x\varphi : \mathbb{R}^n \to \mathbb{R}^3$ *as the linear mappings satisfying:*

$$\forall v \in \mathbb{R}^n, \quad \mathrm{D}^l_x\varphi(v) := \mathrm{Im}\left(\mathrm{D}\tilde{\varphi}_x(v) * \overline{\tilde{\varphi}(x)}\right), \tag{6}$$

and

$$\forall v \in \mathbb{R}^n, \quad \mathrm{D}^r_x\varphi(v) := \mathrm{Im}\left(\overline{\tilde{\varphi}(x)} * \mathrm{D}\tilde{\varphi}_x(v)\right). \tag{7}$$

The following proposition holds:

Proposition 1

$$\forall x \in \mathbb{R}^n, \quad \forall v \in \mathbb{R}^n, \quad \mathrm{D}_x\tilde{\varphi}(v) = \mathrm{D}^l_x\varphi(v) * \varphi(x), \tag{8}$$

and

$$\forall x \in \mathbb{R}^n, \quad \forall v \in \mathbb{R}^n, \quad \mathrm{D}_x\tilde{\varphi}(v) = \varphi(x) * \mathrm{D}^r_x\varphi(v), \tag{9}$$

with the small abuse of notation that in the above formulas, $\mathrm{D}_x\varphi(v)$ *— which belongs to* $\mathbb{R}^3$ *— has to be interpreted as the corresponding purely imaginary quaternion.*

Proof of proposition 1. Since $\varphi(x)$ belongs to Q_u, we can write

$$\forall x \in \mathbb{R}^n, \quad \tilde{\varphi}(x) * \overline{\tilde{\varphi}(x)} = 1. \tag{10}$$

We can differentiate at a given point x, and obtain, by linearity of the conjugate:

$$\forall v \in \mathbb{R}^n, \quad \mathrm{D}\tilde{\varphi}_x(v) * \overline{\tilde{\varphi}(x)} + \tilde{\varphi}(x) * \overline{\mathrm{D}\tilde{\varphi}_x(v)} = 0. \tag{11}$$

We recognize:

$$\forall v \in \mathbb{R}^n, \quad \mathrm{D}\tilde{\varphi}_x(v) * \overline{\tilde{\varphi}(x)} + \overline{\mathrm{D}\tilde{\varphi}_x(v) * \overline{\tilde{\varphi}(x)}} = 0. \tag{12}$$

This means that $\mathrm{D}\tilde{\varphi}_x(v) * \overline{\tilde{\varphi}(x)}$ is a purely imaginary quaternion.

Proposition 2 *The left and right Lie differentials are linked by the following formula:*

$$\forall x \in \mathbb{R}^n, \quad \forall v \in \mathbb{R}^n, \quad \mathrm{D}^r_x\varphi(v) = \overline{\varphi}(x) * \mathrm{D}^l_x\varphi(v) * \varphi(x). \tag{13}$$

In other words,

$$\forall x \in \mathbb{R}^n, \quad \forall v \in \mathbb{R}^n, \quad \mathrm{D}^r_x\varphi(v) = R_{\varphi(x)}\,\mathrm{D}^l_x\varphi(v). \tag{14}$$

Proof of proposition 2. Thanks to formulas (8) and (9):

$$\forall x, \quad \forall v, \quad \mathrm{D}^l_x\varphi(v) * \varphi(x) = \varphi(x) * \mathrm{D}^r_x\varphi(v). \tag{15}$$

Multiplying left by $\overline{\varphi}(x)$ yields:

$$\forall x, \quad \forall v, \quad \overline{\varphi}(x) * \mathrm{D}^l_x\varphi(v) * \varphi(x) = \overline{\varphi}(x) * \varphi(x) * \mathrm{D}^r_x\varphi(v). \tag{16}$$

But, since $\varphi(x)$ is unitary, $\overline{\varphi}(x) * \varphi(x) = (1, 0)$. Hence,

$$\forall x, \quad \forall v, \quad \overline{\varphi}(x) * \mathrm{D}^l_x\varphi(v) * \varphi(x) = \mathrm{D}^r_x\varphi(v), \tag{17}$$

and the proof is done.

3.2 Lie Differentials and Rotation Curves

The above definition can seem arbitrary, but it is not. Loosely speaking, if $\varphi : \mathbb{R}^n \to Q_u$ is a vector-space parameterization of rotations, left — resp., right — Lie differentials of φ are what connects the derivatives in parameter space with angular velocities in the inertial — resp., body — frame. More precisely, let us consider a rotation parameterization $\varphi : \mathbb{R}^n \to Q_u$, and a rotation curve $t \to q(t)$ parameterized by time t, i.e..

$$q(t) := \varphi\big(\alpha(t)\big), \quad \text{where} \quad \begin{cases} t \to \alpha(t) \\ \mathbb{R} \to \mathbb{R}^n \end{cases} \text{is smooth.} \tag{18}$$

If $t \to \omega_I(t)$ is the corresponding angular velocity in the inertial frame, on the one hand, we have — thanks to the quaternion time derivative formula:

$$\dot{q} = \frac{\omega_I}{2} * q. \tag{19}$$

On the other hand, using the chain rule (q is the composition of α by the parameterization φ),

$$\dot{q} = \mathrm{D}_q\tilde{\varphi}\big(\dot{\alpha}\big), \tag{20}$$

and, multiplying — right — by $\overline{q} = \overline{\varphi(\alpha)}$ and using identity (8),

$$\omega_I = 2\,\mathrm{D}_q\tilde{\varphi}(\dot{\alpha}) * \overline{\varphi(\alpha)}. \tag{21}$$

Hence,

$$\omega_I = 2\,\mathrm{D}^l_q\varphi(\dot{\alpha}), \tag{22}$$

Similarly, if $t \to \omega_B(t)$ is the corresponding angular velocity in the body frame, we obtain:

$$\omega_B = 2\,\mathrm{D}^r_q\varphi(\dot{\alpha}). \tag{23}$$

To reiterate, in formulas (22) and (23):

- $t \to q(t)$ is a function whose values are unitary quaternions representing rotations,
- $t \to \omega_I(t)$ — resp., $t \to \omega_B(t)$ — is the corresponding angular velocity in the inertial — resp., body — frame (its values are vectors of $\mathbb{R}^3$),
- $t \to \alpha(t)$ is the parameter curve (its values are vectors of $\mathbb{R}^n$, same for $\dot{\alpha}$),
- $\mathrm{D}^l_q\varphi$ — resp., $\mathrm{D}^r_q\varphi$ — is the left — resp., right — Lie differential of the parameterization at point q. It is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^3$. For a minimal parameterization, $n = 3$, and the differentials can be represented by a 3×3 matrix.

4 Exponential Coordinates

4.1 Definition

The idea behind exponential coordinates is simple: since, geometrically speaking, a 3D rotation is given by a rotation axis and a scalar angle around this axis, we can represent a rotation by a vector $\alpha \in \mathbb{R}^3$, whose direction is the rotation axis, and whose norm is the rotation angle. This corresponds to the quaternion

$$q_\alpha = \left(\cos\left[\tfrac{\|\alpha\|}{2}\right], \sin\left[\tfrac{\|\alpha\|}{2}\right] \frac{\alpha}{\|\alpha\|}\right). \tag{24}$$

Hence, the exponential map e is defined by:

$$e : \begin{cases} \mathbb{R}^3 \to Q_u, \\ \alpha \to \left(\cos\left[\tfrac{\|\alpha\|}{2}\right], \sin\left[\tfrac{\|\alpha\|}{2}\right] \frac{\alpha}{\|\alpha\|}\right). \end{cases} \tag{25}$$

4.2 Lie Differentials of the Exponential Map On Q_u

Proposition 3 (Lie differentials of the exponential map on Q_u — first form) *Let $\alpha \in \mathbb{R}^3$. Then, the left — resp., right — Lie differentials of the exponential map on Q_u $\mathrm{D}^l e_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ — resp., $\mathrm{D}^r e_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ — are given by:*

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^l e_\alpha(v) = \frac{1}{2}\left(\mathrm{sinc}\|\alpha\| v + \left[\frac{1-\cos\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times v + \left[\frac{1-\mathrm{sinc}\,\|\alpha\|}{\|\alpha\|^2}\right] (\alpha \cdot v)\alpha\right), \tag{26}$$

and

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^r e_\alpha(v) = \frac{1}{2}\left(\mathrm{sinc}\|\alpha\| v - \left[\frac{1-\cos\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times v + \left[\frac{1-\mathrm{sinc}\,\|\alpha\|}{\|\alpha\|^2}\right] (\alpha \cdot v)\alpha\right). \tag{27}$$

These formulas can be expressed in another way, which is often more convenient.

Proposition 4 (Lie differentials of the exponential map on Q_u — second form) *Equations (26) and (27) can also be expressed as:*

$$\mathrm{D}^l e_\alpha(v) = \frac{1}{2}\left(v + \left[\frac{1-\cos\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times v + \left[\frac{1-\mathrm{sinc}\,\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times (\alpha \times v)\right), \tag{28}$$

and

$$\mathrm{D}^r e_\alpha(v) = \frac{1}{2}\left(v - \left[\frac{1-\cos\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times v + \left[\frac{1-\mathrm{sinc}\,\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times (\alpha \times v)\right). \tag{29}$$

Moreover, if we denote by $\alpha_\times$ [4] the cross product linear mapping, i.e..

$$\alpha_\times : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ v \to \alpha \times v \end{cases}, \tag{30}$$

[4] Of course, what is hidden here is the Lie algebra $\mathfrak{so}_3$ and its generators…

equations (28) *and* (29) *can be interpreted as an identity between linear mappings, i.e..*

$$\mathrm{D}e^l_\alpha = \frac{1}{2}\left(\mathrm{Id} + f_\alpha \alpha_\times + g_\alpha \alpha_\times^2\right), \quad \textit{and} \quad \mathrm{D}e^r_\alpha = \frac{1}{2}\left(\mathrm{Id} - f_\alpha \alpha_\times + g_\alpha \alpha_\times^2\right), \tag{31}$$

where

$$f_\alpha = \frac{1-\cos\|\alpha\|}{\|\alpha\|^2}, \quad \textit{and } g_\alpha = \frac{1-\operatorname{sinc}\|\alpha\|}{\|\alpha\|^2}. \tag{32}$$

Proof of propositions 3 and 4. We start from definition (25), and differentiate e *as if it were a function* $\tilde{e}$ *from* $\mathbb{R}^3$ *to* $\mathbb{R}^4$. For this we apply the chain rule multiple times. First, notice that the following formula holds for the differential of the norm:

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}_\alpha \|\|(v) = \frac{\alpha \cdot v}{\|\alpha\|}. \tag{33}$$

We deduce:

$$\mathrm{D}\tilde{e}_\alpha(v) = \left(-\sin\left[\tfrac{\|\alpha\|}{2}\right]\frac{\alpha\cdot v}{2\|\alpha\|}, \cos\left[\tfrac{\|\alpha\|}{2}\right]\frac{(\alpha\cdot v)\alpha}{2\|\alpha\|^2} + \sin\left[\tfrac{\|\alpha\|}{2}\right]\frac{v}{\|\alpha\|} - \sin\left[\tfrac{\|\alpha\|}{2}\right]\frac{(\alpha\cdot v)\alpha}{\|\alpha\|^3}\right). \tag{34}$$

Definitions (6) and (7) tell us that we need to compute the imaginary parts of the quaternions $\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}$ and $\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)$. We will also compute their scalar parts: these should be zero, so they can be used as a convenient sanity check. Let us introduce auxiliary quantities which will improve the readability of the computations:

$$\mathrm{c}_\alpha := \cos\left[\tfrac{\|\alpha\|}{2}\right], \quad \text{and} \quad \mathrm{s}_\alpha := \sin\left[\tfrac{\|\alpha\|}{2}\right]. \tag{35}$$

Also, recall that:

$$\overline{e^\alpha} = \left(\cos\left[\tfrac{\|\alpha\|}{2}\right], -\sin\left[\tfrac{\|\alpha\|}{2}\right]\frac{\alpha}{\|\alpha\|}\right). \tag{36}$$

Let us start by considering the left Lie differential, i.e.. the quaternion product $\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}$. Using (34), (36) and the quaternion product formula (4),

$$\left(\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}\right)_s = -s_\alpha c_\alpha \frac{\alpha\cdot v}{2\|\alpha\|} + s_\alpha c_\alpha \frac{(\alpha\cdot v)(\alpha\cdot\alpha)}{2\|\alpha\|^3} + s_\alpha^2 \frac{v\cdot\alpha}{\|\alpha\|^2} - s_\alpha^2\frac{\alpha\cdot v}{\|\alpha\|^2}, \tag{37}$$

that is:

$$\left(\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}\right)_s = 0, \tag{38}$$

as it should be. Now for the imaginary part:

$$\left(\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}\right)_v = s_\alpha^2 \frac{(\alpha\cdot v)\alpha}{2\|\alpha\|^2} + c_\alpha\left[\frac{c_\alpha(\alpha\cdot v)\alpha}{2\|\alpha\|^2} + \frac{s_\alpha v}{\|\alpha\|} - \frac{s_\alpha(\alpha\cdot v)\alpha}{\|\alpha\|^3}\right] - s_\alpha^2\frac{v\times\alpha}{\|\alpha\|^2}. \tag{39}$$

We gather the terms:

$$\left(\mathrm{D}\tilde{e}_\alpha(v) * \overline{e^\alpha}\right)_v = \frac{s_\alpha c_\alpha}{\|\alpha\|}v + \frac{\frac{\|\alpha\|}{2} - s_\alpha c_\alpha}{\|\alpha\|^3}(\alpha\cdot v)\alpha + \frac{s_\alpha^2}{\|\alpha\|^2}\alpha\times v. \tag{40}$$

Hence,

$$\mathrm{D}^l e_\alpha(v) = \frac{s_\alpha c_\alpha}{\|\alpha\|} v + \frac{\frac{\|\alpha\|}{2} - s_\alpha c_\alpha}{\|\alpha\|^3} (\alpha \cdot v)\alpha + \frac{s_\alpha^2}{\|\alpha\|^2} \alpha \times v. \tag{41}$$

Let's work on the scalar coefficients. First,

$$\frac{s_\alpha c_\alpha}{\|\alpha\|} = \frac{1}{\|\alpha\|} \sin\left[\frac{\|\alpha\|}{2}\right] \cos\left[\frac{\|\alpha\|}{2}\right] = \frac{1}{2} \frac{\sin\|\alpha\|}{\|\alpha\|} = \frac{1}{2} \operatorname{sinc}\|\alpha\|. \tag{42}$$

Second, reusing (42),

$$\frac{\frac{\|\alpha\|}{2} - s_\alpha c_\alpha}{\|\alpha\|^3} = \frac{1}{2} \frac{1 - \operatorname{sinc}\|\alpha\|}{\|\alpha\|^2}. \tag{43}$$

Finally,

$$\frac{s_\alpha^2}{\|\alpha\|^2} = \frac{1}{2} \frac{1 - \cos\|\alpha\|}{\|\alpha\|^2}. \tag{44}$$

If we plug (42), (43) and (44) into (41), we get formula (26). Let us now consider the right Lie differential, i.e.. the quaternion product $\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)$. Combining (34), (36) and the quaternion product formula (4),

$$\left(\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)\right)_s = -s_\alpha c_\alpha \frac{\alpha \cdot v}{2\|\alpha\|} + s_\alpha c_\alpha \frac{\|\alpha\|^2 (\alpha \cdot v)}{2\|\alpha\|^3} + s_\alpha^2 \frac{\alpha \cdot v}{\|\alpha\|} - s_\alpha^2 \frac{\|\alpha\|^2 (\alpha \cdot v)}{\|\alpha\|^3}, \tag{45}$$

that is:

$$\left(\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)\right)_s = 0, \tag{46}$$

as it should be. Now for the imaginary part:

$$\left(\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)\right)_v = s_\alpha^2 \frac{(\alpha \cdot v)\alpha}{2\|\alpha\|^2} + c_\alpha \left[\frac{c_\alpha}{2\|\alpha\|^2} (\alpha \cdot v)\alpha + \frac{s_\alpha v}{\|\alpha\|} - \frac{s_\alpha (\alpha \cdot v)\alpha}{\|\alpha\|^3}\right] - \frac{s_\alpha^2}{\|\alpha\|^2} \alpha \times v. \tag{47}$$

Let's gather the terms:

$$\left(\overline{e^\alpha} * \mathrm{D}\tilde{e}_\alpha(v)\right)_v = \frac{s_\alpha c_\alpha}{\|\alpha\|} v + \frac{\frac{\|\alpha\|}{2} - s_\alpha c_\alpha}{\|\alpha\|^3} (\alpha \cdot v)\alpha - \frac{s_\alpha^2}{\|\alpha\|^2} \alpha \times v. \tag{48}$$

Hence,

$$\mathrm{D}^r e_\alpha(v) = \frac{s_\alpha c_\alpha}{\|\alpha\|} v + \frac{\frac{\|\alpha\|}{2} - s_\alpha c_\alpha}{\|\alpha\|^3} (\alpha \cdot v)\alpha - \frac{s_\alpha^2}{\|\alpha\|^2} \alpha \times v, \tag{49}$$

which is the same formula as (41), excepted the third term whose sign is flipped. Formula (27) follows from this observation.

Let us now turn to proving proposition 4. For this, recall the double cross product formula:

$$\forall (u, v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3, \quad u \times (v \times w) = (u \cdot w)v - (u \cdot v)w. \tag{50}$$

Hence,

$$\forall \alpha \in \mathbb{R}^3, \quad \forall v \in \mathbb{R}^3, \quad (\alpha \cdot v)\alpha = \alpha \times (\alpha \times v) + \|\alpha\|^2 v. \tag{51}$$

We deduce:

$$\operatorname{sinc}\|\alpha\| v + \left[\frac{1 - \operatorname{sinc}\|\alpha\|}{\|\alpha\|^2}\right] (\alpha \cdot v)\alpha = v + \left[\frac{1 - \operatorname{sinc}\|\alpha\|}{\|\alpha\|^2}\right] \alpha \times (\alpha \times v). \tag{52}$$

Plugging the above identity into (26) — resp., (27) — gives (28) — resp., (29).

4.3 Lie differentials of the logarithm map on Q_u

The log map is the inverse of the exponential map, i.e..

$$\log : \begin{cases} Q_u \to \mathbb{R}^3, \\ \quad q \to 2\arccos(q_s)\dfrac{q_v}{\|q_v\|}. \end{cases} \tag{53}$$

We *define* the left — resp., right — Lie differential of the log map as the inverse of the left — resp., right — Lie differential of the exponential map: given $q \in Q_u$,

$$\mathrm{D}^l \log_q : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ \quad v \to \left(\mathrm{D}^l e_{\log q}\right)^{-1}(v) \end{cases}, \quad \mathrm{D}^r \log_q : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ \quad v \to \left(\mathrm{D}^r e_{\log q}\right)^{-1}(v) \end{cases} \tag{54}$$

where $\mathrm{D}^l e_{\log q}$ and $\mathrm{D}^r e_{\log q}$ are defined by equation (31), with $\alpha = \log q$. It turns out we can analytically find these inverse mappings.

Proposition 5 (Lie differentials of the logarithm map on Q_u, first form) *Let $q \in Q_u$, and $\alpha = \log q$. The following formulas are true:*

$$\mathrm{D}^l \log_q = 2\left(\mathrm{Id} - \frac{1}{2}\alpha_\times + \left[\frac{1}{\|\alpha\|^2} - \frac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)}\right]\alpha_\times^2\right), \tag{55}$$

and

$$\mathrm{D}^r \log_q = 2\left(\mathrm{Id} + \frac{1}{2}\alpha_\times + \left[\frac{1}{\|\alpha\|^2} - \frac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)}\right]\alpha_\times^2\right) \tag{56}$$

The formulas are sometimes expressed as:

Proposition 6 (Lie differentials of the logarithm map on Q_u, second form) *Let $q \in Q_u$, and $\alpha = \log q$. The following formulas are true:*

$$\mathrm{D}^l \log_q = 2\left(\mathrm{Id} - \frac{1}{2}\alpha_\times + \left[\frac{1}{\|\alpha\|^2} - \frac{1+\cos\|\alpha\|}{2\|\alpha\|\sin\|\alpha\|}\right]\alpha_\times^2\right), \tag{57}$$

and

$$\mathrm{D}^r \log_q = 2\left(\mathrm{Id} + \frac{1}{2}\alpha_\times + \left[\frac{1}{\|\alpha\|^2} - \frac{1+\cos\|\alpha\|}{2\|\alpha\|\sin\|\alpha\|}\right]\alpha_\times^2\right) \tag{58}$$

Yet another variant is:

Proposition 7 (Lie differentials of the logarithm map on Q_u, third form) *Let $q \in Q_u$, and $\alpha = \log q$. The following formulas are true:*

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^l \log_q(v) = 2\left(\frac{\|\alpha\|}{2}\cot\left[\tfrac{\|\alpha\|}{2}\right] v - \frac{1}{2}\alpha \times v + \left[1 - \frac{\|\alpha\|}{2}\cot\left[\tfrac{\|\alpha\|}{2}\right]\right]\frac{\alpha \cdot v}{\|\alpha\|^2}\alpha\right), \tag{59}$$

and

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^r \log_q(v) = 2\left(\frac{\|\alpha\|}{2}\cot\left[\tfrac{\|\alpha\|}{2}\right] v + \frac{1}{2}\alpha \times v + \left[1 - \frac{\|\alpha\|}{2}\cot\left[\tfrac{\|\alpha\|}{2}\right]\right]\frac{\alpha \cdot v}{\|\alpha\|^2}\alpha\right). \tag{60}$$

Proof of propositions 5, 6 and 7. Notice that

$$\alpha_\times^3 = -\|\alpha\|^2 \alpha_\times. \tag{61}$$

Hence, looking at equations (61) and (31) gives us the idea of looking for the inverse mappings under the form

$$\mathrm{D}^l \log_q = 2\left(\mathrm{Id} + \mu_\alpha \alpha_\times + \nu_\alpha \alpha_\times^2\right), \quad \text{and} \quad \mathrm{D}^r \log_q = 2\left(\mathrm{Id} - \mu_\alpha \alpha_\times + \nu_\alpha \alpha_\times^2\right), \tag{62}$$

where

$$\alpha = \log q,\ \mu_\alpha : \mathbb{R} \to \mathbb{R},\ \text{and}\ \nu_\alpha : \mathbb{R} \to \mathbb{R}. \tag{63}$$

Let us find the unknowns μ_α and ν_α. On the one hand, we must have $\mathrm{D}^l e_\alpha \, \mathrm{D}^l \log_q = \mathrm{Id}$, which translates to:

$$\left(\mathrm{Id} + \mu_\alpha \alpha_\times + \nu_\alpha \alpha_\times^2\right)\left(\mathrm{Id} + f_\alpha \alpha_\times + g_\alpha \alpha_\times^2\right) = \mathrm{Id}, \tag{64}$$

where f_α and g_α are defined by (32). Thanks to (61), the left hand side only has terms in Id, $\alpha_\times$ and $\alpha_\times^2$, and the previous equation becomes:

$$\left(f_\alpha + \mu_\alpha - \|\alpha\|^2 \mu_\alpha g_\alpha - \|\alpha\|^2 \nu_\alpha f_\alpha\right)\alpha_\times + \left(g_\alpha + \mu_\alpha f_\alpha + \nu_\alpha - \|\alpha\|^2 \nu_\alpha g_\alpha\right)\alpha_\times^2 = 0. \tag{65}$$

We thus have to solve the following 2×2 linear system in μ_α and ν_α:

$$\begin{cases} (1 - \|\alpha\|^2 g_\alpha)\mu_\alpha - \|\alpha\|^2 f_\alpha \quad \nu_\alpha = & -f_\alpha, \\ f_\alpha \mu_\alpha + (1 - \|\alpha\|^2 g_\alpha) \quad \nu_\alpha = & -g_\alpha. \end{cases} \tag{66}$$

On the other hand, we must have $\mathrm{D}^r e_\alpha \, \mathrm{D}^r \log_q = \mathrm{Id}$, which translates to:

$$\left(\mathrm{Id} - \mu_\alpha \alpha_\times + \nu_\alpha \alpha_\times^2\right)\left(\mathrm{Id} - f_\alpha \alpha_\times + g_\alpha \alpha_\times^2\right) = \mathrm{Id}. \tag{67}$$

If we expand this identity, using again (61), we obtain:

$$\left(-f_\alpha - \mu_\alpha + \|\alpha\|^2 \mu_\alpha g_\alpha + \|\alpha\|^2 \nu_\alpha f_\alpha\right)\alpha_\times + \left(g_\alpha + \mu_\alpha f_\alpha + \nu_\alpha - \|\alpha\|^2 \nu_\alpha g_\alpha\right)\alpha_\times^2 = 0, \tag{68}$$

which also corresponds to linear system (66). Solving this system yields:

$$\begin{cases} \mu_\alpha = & \dfrac{-\|\alpha\|^2 f_\alpha g_\alpha - (1 - \|\alpha\|^2 g_\alpha) f_\alpha}{\|\alpha\|^2 f_\alpha^2 + (1 - \|\alpha\|^2 g_\alpha)^2}, \\ \nu_\alpha = & \dfrac{f_\alpha^2 - (1 - \|\alpha\|^2 g_\alpha) g_\alpha}{\|\alpha\|^2 f_\alpha^2 + (1 - \|\alpha\|^2 g_\alpha)^2}, \end{cases} \tag{69}$$

that is:

$$\begin{cases} \mu_\alpha = \dfrac{-f_\alpha}{\|\alpha\|^2 f_\alpha^2 + (1 - \|\alpha\|^2 g_\alpha)^2}, \\ \nu_\alpha = \dfrac{f_\alpha^2 - (1 - \|\alpha\|^2 g_\alpha) g_\alpha}{\|\alpha\|^2 f_\alpha^2 + (1 - \|\alpha\|^2 g_\alpha)^2}. \end{cases} \tag{70}$$

We can now replace f_α and g_α by their formulas, to simplify things a little bit:

$$\|\alpha\|^2 f_\alpha^2 + (1-\|\alpha\|^2 g_\alpha)^2 = \frac{1}{\|\alpha\|^2}(1-\cos\|\alpha\|)^2 + \operatorname{sinc}^2\|\alpha\|^2 = \frac{2(1-\cos\|\alpha\|)}{\|\alpha\|^2} = 2f_\alpha. \quad (71)$$

Also,

$$f_\alpha^2 - (1-\|\alpha\|^2 g_\alpha)g_\alpha = \frac{1}{\|\alpha\|^4}\left((1-\cos\|\alpha\|)^2 - \sin\|\alpha\|(\|\alpha\| - \sin\|\alpha\|)\right), \quad (72)$$

which translates to

$$f_\alpha^2 - (1-\|\alpha\|^2 g_\alpha)g_\alpha = \frac{1}{\|\alpha\|^4}\left(2(1-\cos\|\alpha\|) - \|\alpha\|\sin\|\alpha\|\right) = \frac{1}{\|\alpha\|^2}(2f_\alpha - \operatorname{sinc}\|\alpha\|). \quad (73)$$

Putting (70), (71) and (73) together:

$$\begin{cases} \mu_\alpha = -\dfrac{1}{2}, \\ \nu_\alpha = \dfrac{1}{\|\alpha\|^2} - \dfrac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)}, \end{cases} \quad (74)$$

which, combined with (62), gives formulas (55) and (56), i.e.. proposition 5. Since

$$\frac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)} = \frac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)}\frac{1+\cos\|\alpha\|}{1+\cos\|\alpha\|} = \frac{\sin\|\alpha\|(1+\cos\|\alpha\|)}{2\|\alpha\|(1-\cos^2\|\alpha\|)} = \frac{1+\cos\|\alpha\|}{2\|\alpha\|\sin\|\alpha\|}, \quad (75)$$

Proposition 6 also holds. Finally, let's prove proposition 7. First, notice that:

$$\frac{\sin\|\alpha\|}{2\|\alpha\|(1-\cos\|\alpha\|)} = \frac{1}{2\|\alpha\|}\frac{2\sin\left[\frac{\|\alpha\|}{2}\right]\cos\left[\frac{\|\alpha\|}{2}\right]}{2\sin^2\left[\frac{\|\alpha\|}{2}\right]} = \frac{1}{2\|\alpha\|}\cot\left[\frac{\|\alpha\|}{2}\right]. \quad (76)$$

Also,

$$\forall v \in \mathbb{R}^3, \quad \alpha_\times^2 v = (\alpha\cdot v)\alpha - \|\alpha\|^2 v. \quad (77)$$

Thus,

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^l \log_q(v) = 2\left(v - \frac{1}{2}\alpha\times v + \left[\frac{1}{\|\alpha\|^2} - \frac{1}{2\|\alpha\|}\cot\left[\frac{\|\alpha\|}{2}\right]\right]\left((\alpha\cdot v)\alpha - \|\alpha\|^2 v\right)\right), \quad (78)$$

and:

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^l \log_q(v) = 2\left(\frac{\|\alpha\|}{2}\cot\left[\frac{\|\alpha\|}{2}\right]v - \frac{1}{2}\alpha\times v + \left[1 - \frac{\|\alpha\|}{2}\cot\left[\frac{\|\alpha\|}{2}\right]\right]\frac{\alpha\cdot v}{\|\alpha\|^2}\alpha\right), \quad (79)$$

which is (59). Similarly, (60) holds. Formulas (59) and (60) are probably the most efficient, implementation wise. Formula (60) is the one derived in [3].

4.4 Advantages and drawbacks

Advantages

- Exponential coordinates are a minimal representation of rotations.
- This representation is quite intuitive, and the geometrical parameters of a rotation (axis and angle) are easily extracted from its quaternion representation.
- Since this parametrization is unconstrained and lies in a vector space, it can be used "as-is" in any optimization algorithm using derivatives or Jacobians (Gauss-Newton, L-BFGS, etc.).

Drawbacks

- Exponential coordinates have singularities — as do every minimal parameterization of a true manifold —, which can render them unsuitable for some applications. However, these singularities are less restrictive than Euler angles' gimbal locks.
- Exponential coordinate Jacobians are quite costly to compute, in particular because their formulas involve a lot of transcendental functions.

5 Modified Rodrigues Parameters

As seen in the previous section, exponential coordinates encode a rotation as a 3D vector, whose direction is the rotation axis, and norm is the rotation angle. Of course, many alternative 3D parameterizations could be devised, by encoding a rotation as a 3D vector whose direction is the rotation axis, and norm is any strictly increasing function of the rotation angle. Among these, the *Modified Rodrigues Parameters* — MRPs —([4]) are a very interesting parameterization, which, compared with exponential coordinates, offers a somewhat better handling of singularities, and much cheaper differential evaluation. Geometrically, this parameterization corresponds to the 4D stereographic projection of Q_u — seen as the unit ball of $\mathbb{R}^4$ — on a 3D hyperplane.

5.1 Definition

Analytically, if we consider a rotation of normalized axis n and angle θ, we represent it by the 3D vector

$$\beta := \tan\left[\tfrac{\theta}{4}\right] n. \tag{80}$$

It turns out the quaternionic representation of our rotation can be expressed as a rational fraction of β and $\|\beta\|$. To see this, recall the classical trigonometric expressions:

$$\forall x \in [0, 2\pi[, \quad \cos 2x = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad \sin 2x = \frac{2t}{1+t^2}, \quad \text{where} \quad t := \tan\left[\tfrac{x}{2}\right]. \tag{81}$$

Since the quaternionic representation of the rotation is

$$q = \left(\cos\left[\tfrac{\theta}{2}\right], \sin\left[\tfrac{\theta}{2}\right] n\right), \tag{82}$$

we deduce its quaternionic representation as a function of β:

$$q = \left(\frac{1-\|\beta\|^2}{1+\|\beta\|^2}, \frac{2\beta}{1+\|\beta\|^2}\right). \tag{83}$$

5.2 Lie Differentials of the MRP Mapping

Proposition 8 *Consider the MRP parameterization* $m : \mathbb{R}^3 \to Q_u$*, defined as:*

$$m(\beta) = \left(\frac{1-\|\beta\|^2}{1+\|\beta\|^2}, \frac{2\beta}{1+\|\beta\|^2}\right). \tag{84}$$

Then, its left — resp., right — Lie differential $\mathrm{D}^l m_\beta$ *— resp.,* $\mathrm{D}^r m_\beta$ *— are given by the following formulas:*

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^l m_\beta(v) = [1 + m_s(\beta)]\left(v + m_v(\beta)\times v\right) + m_v(\beta)\times(m_v(\beta)\times v), \tag{85}$$

and

$$\forall v \in \mathbb{R}^3, \quad \mathrm{D}^r m_\beta(v) = [1 + m_s(\beta)]\left(v - m_v(\beta)\times v\right) + m_v(\beta)\times(m_v(\beta)\times v). \tag{86}$$

Proof of proposition 8. As usual, we start by computing the differential of the MRP mapping m *considered as a mapping* $\tilde{m}$ *from* $\mathbb{R}^3$ *to* $\mathbb{R}^4$. Let us introduce the auxiliary functions

$$\varphi : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \to \dfrac{1-t}{1+t} \end{cases}, \quad \text{and} \quad \psi : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \to \dfrac{2}{1+t} \end{cases}. \tag{87}$$

We have

$$\tilde{m}(\beta) = \left(\varphi\left(\|\beta\|^2\right), \psi\left(\|\beta\|^2\right)\beta\right). \tag{88}$$

Hence, using the chain rule:

$$\mathrm{D}\tilde{m}_\beta(v) = \Big(2\varphi'(\|\beta\|^2)(\beta\cdot v),\ \psi(\|\beta\|^2)v + 2\psi'(\|\beta\|^2)(\beta\cdot v)\beta\Big). \tag{89}$$

Since

$$\varphi'(t) = -\frac{2}{(1+t)^2}, \quad \text{and} \quad \psi'(t) = -\frac{2}{(1+t)^2}, \tag{90}$$

we deduce:

$$\mathrm{D}\tilde{m}_\beta(v) = \left(-\frac{4(\beta\cdot v)}{(1+\|\beta\|^2)^2}, \frac{2v}{1+\|\beta\|^2} - \frac{4(\beta\cdot v)\beta}{(1+\|\beta\|^2)^2}\right). \tag{91}$$

This can be expressed as an expression of $m(\beta)$ components only. To see this, notice that:

$$\frac{2}{1+\|\beta\|^2} = 1 + \frac{1-\|\beta\|^2}{1+\|\beta\|^2} = 1 + m_s(\beta), \tag{92}$$

and, consequently,

$$\frac{4(\beta\cdot v)}{(1+\|\beta\|^2)^2} = \frac{2}{1+\|\beta\|^2}\frac{\beta\cdot v}{1+\|\beta\|^2} = \big[1 + m_s(\beta)\big]m_v(\beta)\cdot v. \tag{93}$$

On the other hand:

$$\frac{4(\beta\cdot v)\beta}{(1+\|\beta\|^2)^2}=\frac{2(\beta\cdot v)}{1+\|\beta\|^2}\frac{2\beta}{1+\|\beta\|^2}=(m_v(\beta)\cdot v)m_v(\beta). \tag{94}$$

Hence, equation (91) becomes:

$$\mathrm{D}\tilde{m}_\beta(v)=\big(-\big[1+m_s(\beta)\big]\big(m_v(\beta)\cdot v\big),\ \big[1+m_s(\beta)\big]v-(m_v(\beta)\cdot v)m_v(\beta)\big)\,. \tag{95}$$

One can marvel at the simplicity of this equation, especially when comparing it with (34). We can now deduce formulas (85) and (86). Recall that:

$$\overline{m}(\beta)=(m_s(\beta),\ -m_v(\beta))\,. \tag{96}$$

On the one hand, let us verify, as a sanity check, that $\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)$ and $\overline{m}(\beta)*\mathrm{D}\tilde{m}_\beta(v)$ are purely imaginary. Thanks to quaternion product definition (4), we have

$$\begin{aligned}&\left(\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)\right)_s=\\&-m_s(\beta)\big[1+m_s(\beta)\big](m_v(\beta)\cdot v)+\big[1+m_s(\beta)\big](m_v(\beta)\cdot v)-\|m_v(\beta)\|^2(m_v(\beta)\cdot v),\end{aligned} \tag{97}$$

that is:

$$\left(\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)\right)_s=\left[1-m_s^2(\beta)-\|m_v(\beta)\|^2\right](m_v(\beta)\cdot v). \tag{98}$$

Finally,

$$\left(\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)\right)_s=0, \tag{99}$$

as it should be. Since $\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)=\overline{m}(\beta)*\mathrm{D}\tilde{m}_\beta(v)$, we also have

$$\left(\overline{m}(\beta)*\mathrm{D}\tilde{m}_\beta(v)\right)_s=0. \tag{100}$$

On the other hand, let us compute the vector parts of $\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)$ and $\overline{m}(\beta)*\mathrm{D}\tilde{m}_\beta(v)$. Thanks to (4), we have

$$\begin{aligned}\left(\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)\right)_v=&\left[1+m_s(\beta)\right](m_v(\beta)\cdot v)m_v(\beta)+m_s(\beta)\left[1+m_s(\beta)\right]v-\\&m_s(\beta)(m_v(\beta)\cdot v)m_v(\beta)-\left[1+m_s(\beta)\right]v\times m_v(\beta),\end{aligned} \tag{101}$$

that is

$$\begin{aligned}&\left(\mathrm{D}\tilde{m}_\beta(v)*\overline{m}(\beta)\right)_v=\\&\left[1+m_s(\beta)\right]\Big(m_s(\beta)v+(m_v(\beta)\cdot v)m_v(\beta)-v\times m_v(\beta)\Big)-m_s(\beta)(m_v(\beta)\cdot v)m_v(\beta).\end{aligned} \tag{102}$$

We recognize:

$$\mathrm{D}^l m_\beta(v)=m_s(\beta)\left[1+m_s(\beta)\right]v+\left[1+m_s(\beta)\right]m_v(\beta)\times v+(m_v(\beta)\cdot v)m_v(\beta). \tag{103}$$

Now, thanks to the double cross product formula,

$$(m_v(\beta)\cdot v)m_v(\beta) = m_v(\beta)\times(m_v(\beta)\times v) + \|m_v(\beta)\|^2 v. \tag{104}$$

From this, combined with the fact that $m_s^2(\beta) + \|m_v(\beta\|^2 = 1$, we deduce:

$$\mathrm{D}^l m_\beta(v) = [1+m_s(\beta)]\,\big(v + m_v(\beta)\times v\big) + m_v(\beta)\times(m_v(\beta)\times v), \tag{105}$$

i.e.. formula (85). Similarly, by definition of the quaternion product (4),

$$\begin{gathered}(\overline{m}(\beta) * \mathrm{D}\tilde{m}_\beta(v))_v = \\ [1+m_s(\beta)]\left(m_s(\beta)v + (m_v(\beta)\cdot v)m_v(\beta) + v\times m_v(\beta)\right) - m_s(\beta)(m_v(\beta)\cdot v)m_v(\beta),\end{gathered} \tag{106}$$

and

$$\mathrm{D}^r m_\beta(v) = m_s(\beta)\,[1+m_s(\beta)]\,v - [1+m_s(\beta)]\,m_v(\beta)\times v + (m_v(\beta)\cdot v)m_v(\beta). \tag{107}$$

Using — again — the double product formula, we deduce

$$\mathrm{D}^r m_\beta(v) = [1+m_s(\beta)]\,\big(v - m_v(\beta)\times v\big) + m_v(\beta)\times(m_v(\beta)\times v), \tag{108}$$

i.e.. formula (86).

5.3 Inverse MRP Mapping

The inverse MRP mapping is to the MRP mapping what the logarithm map is to the exponential map (see section 4). Recall that, by definition, the MRP mapping m is defined by:

$$m : \begin{cases} \mathbb{R}^3 \to Q_u, \\ \beta \to \left(\dfrac{1-\|\beta\|^2}{1+\|\beta\|^2}, \dfrac{2\beta}{1+\|\beta\|^2}\right). \end{cases} \tag{109}$$

Proposition 9 *The reverse* MRP *mapping is given by*

$$m^{-1} : \begin{cases} Q_u \to \mathbb{R}^3, \\ q \to \dfrac{q_v}{1+q_s}. \end{cases} \tag{110}$$

Proof of proposition 9. We need to invert formula (109), i.e..

$$\begin{cases} q_s = \dfrac{1-\|\beta\|^2}{1+\|\beta\|^2}, \\ q_v = \dfrac{2\beta}{1+\|\beta\|^2}. \end{cases} \tag{111}$$

From the first equation, we deduce:

$$\|\beta\|^2 = \frac{1-q_s}{1+q_s}. \tag{112}$$

Hence,

$$\frac{2}{1+\|\beta\|^2} = \frac{2}{\frac{2}{1+q_s}} = 1+q_s. \tag{113}$$

If we apply the previous identity to the second equation of (111), we have

$$q_v = \beta(1+q_s). \tag{114}$$

Hence,

$$\beta = \frac{q_v}{1+q_s}, \tag{115}$$

and the proof is done. Again, we can marvel at the simplicity of this formula, compared to (53).

5.4 Inverse MRP Mapping Differentials

Let us define the left and right — MRP inverse mapping differentials $\mathrm{D}^l m^{-1}$ and $\mathrm{D}^r m^{-1}$ as the inverse of the left and right MRP mapping differentials $\mathrm{D}^l m$ and $\mathrm{D}^r m$.

Proposition 10 *Let $q = (q_s, q_v)$ be an element of Q_u. Let us define by $q_\times$ the cross product linear mapping with the vector part of q, i.e..*

$$q_\times : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3, \\ q \to q_v \times v. \end{cases} \tag{116}$$

The inverse MRP *mapping differentials around q are then given by the following formulas:*

$$\mathrm{D}^l_q m^{-1} = \tfrac{1}{1+q_s}\,\mathrm{Id} - \beta_l(q) q_\times + \gamma_l(q) q_\times^2, \tag{117}$$

where

$$\begin{cases} \beta_l(q) = \dfrac{1+q_s}{\|q_v\|^2(1+q_s)^2 + (1+q_s-\|q_v\|^2)^2}, \\ \gamma_l(q) = \dfrac{1}{\|q_v\|^2(1+q_s)^2 - (1+q_s-\|q_v\|^2)^2}, \end{cases} \tag{118}$$

and

$$\mathrm{D}^r_q q^{-1} = \tfrac{1}{1+q_s}\,\mathrm{Id} + \beta_r(q) q_\times + \gamma_r(q) q_\times^2, \tag{119}$$

where

$$\begin{cases} \beta_r(q) = \dfrac{1+q_s}{\|q_v\|^2(1+q_s)^2 + (1+q_s-\|q_v\|^2)^2}, \\ \gamma_r(q) = \dfrac{1}{\|q_v\|^2(1+q_s)^2 + (1+q_s-\|q_v\|^2)^2}. \end{cases} \tag{120}$$

Proof of proposition 10. Let us start with equation (117). Notice that the identity giving the left MRP differential (85) can be expressed as an identity between linear mappings:

$$\mathrm{D}^l_\beta m = (1+q_s)\big(\mathrm{Id} + q_\times\big) + q_\times^2, \tag{121}$$

where $q = m(\beta)$. This gives the idea of looking for the inverse on the same form, i.e..

$$\mathrm{D}^l_q m^{-1} = \alpha(q)\,\mathrm{Id} + \beta(q) q_\times + \gamma(q) q_\times^2. \tag{122}$$

Since we must have

$$\mathrm{Id} = \mathrm{D}^l_q m^{-1}\big(\mathrm{D}^l_\beta m\big), \tag{123}$$

we can deduce the equations satisfied by α, β and γ. Recall that

$$q_\times^3 = -\|q\|^2 q_\times, \tag{124}$$

so, expanding the right hand side of (123) yields

$$\begin{aligned} \mathrm{Id} = \quad & \alpha(q)\Big((1+q_s)\big(\mathrm{Id} + q_\times\big) + q_\times^2\Big) + \\ & \beta(q)\Big((1+q_s)\big(q_\times + q_\times^2\big) - \|q_v\|^2 q_\times\Big) + \\ & \gamma(q)\Big((1+q_s)\big(q_\times^2 - \|q_v\|^2 q_\times\big) - \|q_v\|^2 q_\times^2\Big). \end{aligned} \tag{125}$$

Hence,

$$\begin{aligned} \mathrm{Id} = \quad & \alpha(q)(1+q_s)\,\mathrm{Id} + \\ & \Big(\alpha(q)(1+q_s) + \beta(q)\big(1+q_s-\|q_v\|^2\big) - \gamma(q)\|q_v\|^2(1+q_s)\Big) q_\times + \\ & \Big(\alpha(q) + \beta(q)(1+q_s) + \gamma(q)\big(1+q_s-\|q_v\|^2\big)\Big) q_\times^2. \end{aligned} \tag{126}$$

Introducing

$$a(q) := 1+q_s, \quad \text{and} \quad b(q) := 1+q_s-\|q_v\|^2, \tag{127}$$

we deduce

$$\begin{aligned} \mathrm{Id} = \quad & a(q)\alpha(q)\,\mathrm{Id} + \\ & \Big(a(q)\alpha(q) + b(q)\beta(q) - a(q)\|q_v\|^2\gamma(q)\Big) q_\times + \\ & \Big(a(q)\alpha(q) + a(q)^2\beta(q) + a(q)b(q)\gamma(q)\Big) q_\times^2. \end{aligned} \tag{128}$$

Since Id, $q_\times$ and $q_\times^2$ are linearly independent, we obtain:

$$\begin{cases} a\alpha & = 1, \\ a\alpha + b\beta & -a\|q_v\|^2\gamma = 0, \\ a\alpha + a^2\beta & +ab\gamma = 0. \end{cases} \tag{129}$$

Note that, in the above equation, we dropped the dependency in q for terseness. We deduce

$$a(q) = \frac{1}{\alpha(q)}, \tag{130}$$

and

$$\begin{cases} b\beta - a\|q_v\|^2\gamma & = -1, \\ a^2\beta \qquad + ab\gamma & = -1. \end{cases} \tag{131}$$

Solving (131) in β and γ yields:

$$\begin{cases} \beta = \dfrac{-b - \|q_v\|^2}{b^2 + \|q_v\|^2 a^2}, \\ \gamma = \dfrac{b - a^2}{ab^2 - \|q_v\|^2 a^3}. \end{cases} \tag{132}$$

Now notice that

$$-b - \|q_v\|^2 = -(a - \|q_v\|^2) - \|q_v\|^2 = -a, \tag{133}$$

and

$$\begin{aligned} b - a^2 &= (1 + q_s) - \|q_v\|^2 - (1 + q_s)^2 \\ &= -q_s - q_s^2 - \|q_v\|^2 \\ &= -1 - q_s \\ &= -a. \end{aligned} \tag{134}$$

Finally,

$$\begin{cases} \beta = \dfrac{-a}{\|q_v\|^2 a^2 + b^2}, \\ \gamma = \dfrac{1}{\|q_v\|^2 a^2 - b^2}, \end{cases} \tag{135}$$

and we recover formula (117). for formula (119), we start from identity

$$\mathrm{Id} = \mathrm{D}^r_q m^{-1}\big(\mathrm{D}^r_\beta m\big). \tag{136}$$

Similar computations as before yield the following system in α, β and γ:

$$\begin{cases} a\alpha & = 1, \\ -a\alpha + b\beta \quad + a\|q_v\|^2\gamma & = 0, \\ a\alpha - a^2\beta \qquad + ab\gamma & = 0. \end{cases} \tag{137}$$

As before,

$$a(q) = \frac{1}{1 + q_s}, \tag{138}$$

and, this time,

$$\begin{cases} b\beta + a\|q_v\|^2\gamma & = 1, \\ -a^2\beta \qquad + ab\gamma & = -1. \end{cases} \tag{139}$$

Solving (139) in β and γ yields:

$$\begin{cases} \beta = \dfrac{b + \|q_v\|^2}{b^2 + \|q_v\|^2 a^2}, \\ \gamma = \dfrac{a^2 - b}{ab^2 + \|q_v\|^2 a^3}. \end{cases} \tag{140}$$

Reusing identities and , we deduce

$$\begin{cases} \beta = \dfrac{a}{\|q_v\|^2 a^2 + b^2}, \\ \gamma = \dfrac{1}{\|q_v\|^2 a^2 + b^2}, \end{cases} \tag{141}$$

which proves (119).

5.5 Advantages and Drawbacks

The MRP parameterization has the same advantages and drawbacks as the exponential coordinates, except that:

- its Lie differentials are much simpler, and consequently, much more efficient to compute.
- its formulas are always valid, except for the inverse mapping around singularities corresponding to $q_s = -1$, i.e. angles of 180^o. Conversely, exponential map formulas are all formally singular for small angles, and thus have to be replaced by Taylor approximations, which yields lesser performance. Also, extra care is needed to set a proper threshold to trigger the switch to these non-singular alternate expressions.

Changelog

- 2023-04-21: Initial release. Thanks to Alexander Clarke for his careful proofreading of the manuscript.

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